Shot Noise in Gravitational-Wave Detectors with Fabry-Perot Arms

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The shot noise limited sensitivity is calculated for gravitational-wave interferometers with Fabry-Perot arms, similar to those being installed at the Laser Interferometer Gravitational-wave Observatory (LIGO) and the VIRGO facility. This calculation includes the effect of nonstationary shot noise due to phase modulation of the light. The resulting formula is experimentally verified using a test interferometer with 40-meter arms.

I Introduction

Interferometric gravitational wave detectors with multiple-kilometer baselines are currently under construction by the LIGO project in the US and the VIRGO project in Italy. Interferometers with baselines of several hundred meters are under construction in the GEO600 projects in Germany and the TAMA project in Japan. These kilometer-scale detectors will be sensitive to relative displacements of their test masses of $10^{-21}$ m/$\sqrt{\text{Hz}}$ in the frequency band from approximately 10 Hz to several 1000 Hz. At frequencies above approximately 300 Hz the dominant noise source is expected to be photon shot noise.

The sensitivity limit imposed by photon shot noise depends on the optical configuration of the interferometer as well as the technique employed to read out the relative positions of the test masses. There has been considerable effort devoted to exploring novel optical configurations and readout schemes that improve the shot noise limited performance of the detectors for a given laser power without requiring unreasonably high power levels in the interferometer.\textsuperscript{5,6,7}

The optical configuration selected for the initial LIGO and VIRGO detectors is a power recycled interferometer with Fabry-Perot arm cavities\textsuperscript{8,9} as shown in Figure 1. A passing gravitational wave incident from directly overhead will produce a fluctuating strain, that stretches one arm of the interferometer and contracts the other arm for half of the gravitational wave period. The interferometer measures the change in the difference between the arm lengths in a way directly analogous to a Michelson interferometer. The Fabry-Perot cavities are held on resonance by length control servos and the beam splitter is controlled so that the light returning from the two arms interferes destructively at the antisymmetric port. This light interferes constructively at the symmetric port, with light returning toward the laser in accordance with energy conservation. The very small deviation from resonance induced by a passing gravitational wave will cause the phase of the light reflected from the arms to change, spoiling the destructive interference at the antisymmetric port. This gives rise to the gravitational wave signal. The recycling mirror improves the shot noise limited sensitivity by redirecting the light returning to the laser back into the interfer-
ometer. The recycling mirror must be positioned so that the light it reflects back into the interferometer interferes constructively with the light transmitted through it from the laser.

Calculating the shot noise limited sensitivity of a gravitational wave interferometer is complicated by the fact that to achieve adequate sensitivity, the light in the interferometer is phase modulated. The output light power is time varying at the phase modulation frequency and its harmonics. Thus, the associated shot noise is nonstationary. Early treatments assumed that the shot noise was a white noise source with a variance proportional to the time-averaged power incident on the photo-detector. This approximation is useful for making order of magnitude predictions of shot noise limited sensitivity, but more accurate comparisons with experiment require including the effect of phase modulation and the demodulation waveform used. This has been done for a single Fabry-Perot cavity and for a delay line interferometer and the dependence of the shot noise on the demodulation waveform has been experimentally confirmed. A shot noise limited optical phase measurement has also been demonstrated at high optical power in a power recycled Michelson Interferometer.

Fabry-Perot cavities are used in the arms of a power recycled Michelson interferometer to provide a large amplification of the optical phase shift generated by a gravitational wave. Here we derive the shot noise limited sensitivity for such a power-recycled interferometer with Fabry-Perot arm cavities which includes the effect of phase modulation applied to the light incident on the interferometer. The resulting formula is directly compared with data from a 40-m-long, suspended-mirror, test interferometer. The theory and experimental data are in good agreement within experimental uncertainties.

In Section II, we derive the response of the interferometer to mirror displacements. The power spectrum of shot noise at the demodulated signal output of the interferometer is given in Section III. Section IV describes an experimental confirmation of the calculated shot noise contribution to a test interferometer that uses Fabry-Perot arms with a 40-meter baseline.

II Interferometer Response

A recycled interferometer with the mirrors and fields labeled is shown in Figure 2. We specifically derive the contribution of shot noise in an interferometer that uses phase-modulation on the incident light. However the technique employed in this work is generally applicable to other configurations that apply phase modulation. The light incident from the laser is \( E_0 \). The light is phase modulated, with modulation depth \( \Gamma \), between the laser and the interferometer. This impresses sidebands on the light at frequencies above and below the laser frequency (carrier), separated by the modulation frequency and its harmonics. The carrier light leaving the antisymmetric port is \( E_A \), which is typically very small in the absence of a signal because the antisymmetric port is held on a dark fringe for the carrier. Because of the asymmetry, the sidebands are not on a dark fringe at the antisymmetric port. We adopt the phase convention that the first order sidebands have real amplitudes of opposite sign when incident on the beam splitter. The second order sidebands have equal real amplitudes of the same sign. We will neglect terms in the calculation of order \( \Gamma^3 \) because the modulation depth is assumed small. Thus, we will only need to consider up to second order sidebands. The transmission of the \( n \)-th order sideband from incidence on the beam splitter to the antisymmetric port is \( \pm i \sin n\alpha \), where positive indicates the upper sideband.
and negative, the lower sideband. The amplitude of the total complex field at the antisymmetric port is

\[ E_{\text{anti}} = E_A + iE_+ e^{i\omega t} + iE_+ e^{-i\omega t} + iE_{2+} e^{2i\omega t} - iE_{2+} e^{-2i\omega t} \]  

(1)

where \( \omega \) is the angular modulation frequency, and \( E_+ \) and \( E_{2+} \) are the magnitudes of the first order and second order sideband fields at the antisymmetric port.

The detection system is modeled as a photodetector, a demodulator and a low-pass filter as shown in Figure 3. The low-pass filter need not be explicitly built as a separate element following the mixer output. In practice, all of the servo loops that derive their error signals from the mixer output have unity gain frequencies that are low compared to the modulation frequency. Thus, we shall ignore in our analysis any signals at the mixer output that are at or above the modulation frequency.

A gravitational wave interacting with the detector will produce the same differential mode signal as shaking mirror 4 by some other means. If we displace mirror 4 such that \( x_4 = x_0 \sin \Omega t \), for sufficiently small \( x_0 \) this will produce a signal at the antisymmetric port given by:

\[ E_A = E_{\text{DC}} - i k E_2 \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin (\Omega t + \psi)}{\sqrt{1 + (\Omega / \omega_c)^2}} \]  

(2)

where \( k \) is the wave number of the light, \( E_{\text{DC}} \) is the field due to the contrast defect which comes from any non-interfering light on the photodetector and \( \psi \) is a phase factor irrelevant to this analysis. \( \omega_c \) is the angular frequency of the so-called cavity pole,

\[ \omega_c = \frac{c}{2l} \frac{1 - r_3 r_4}{r_3 r_4} \]  

(3)

whose value is typically within the bandwidth of interest for gravitational waves. (These equations are derived in greater detail in Appendix A.) The modulation sidebands do not resonate in the arm cavities and thus are not affected by the motion of mirror 4.

Expressing the fields in units of \( \text{photoelectrons/second} \), simplifies the following formulae in our analysis. The photocurrent, \( i_p \), is

\[ i_p = \frac{1}{2} \left( e^{2i(k \pm nK)l_1} - e^{-2i(k \pm nK)l_2} \right) \]

where \( K \) equals the wavenumber at the modulation frequency. Let \( l = \frac{1}{2}(l_1 + l_2) \). Then, neglecting unimportant phase factors and accounting for the carrier being on a dark fringe,

\[ i_p = \frac{1}{2} \left( e^{2i(k + nK)l + \delta/2} - e^{-2i(k + nK)l - \delta/2} \right) \]

\[ = \frac{1}{2} \left( e^{i nK \delta} - e^{-i nK \delta} \right) \]

\[ = \pm i \sin n \alpha \]
The photocurrent has components at DC, $\omega$ and $2\omega$. The effect of the mixer and low pass filter is to pick out the $\omega$ component, which is

$$i_\rho = |E_A + iE_+ e^{i\omega t} + iE_+ e^{-i\omega t} + iE_2 + e^{2i\omega t} - iE_2 - e^{-2i\omega t}|^2$$

$$= |E_A|^2 + 2E_+^2 + 4E_4 \mathop{\text{Im}}[E_A] \cos \omega t - 2E_+^2 \cos 2\omega t - 4E_2 \mathop{\text{Re}}[E_A] \sin 2\omega t$$

The effect of the mixer and low pass filter is to pick out the $\omega$ component, which is

$$4kE_2^2 E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin (\Omega t + \psi)}{\sqrt{1 + (\Omega / \omega_c)^2}} \cos \omega t$$

Because the photocurrent is modulated, it is important to treat the shot noise as a nonstationary random process and to consider the actual demodulation waveform used.

The effective demodulation waveform used in the 40-m interferometer is cosinusoidal. Square wave demodulation is used at the mixer, but the band-pass filter, which is built into the photodiode and centered on the modulation frequency, makes this effectively cosinusoidal demodulation. This is because the square wave can be decomposed into a sum of cosine waves at odd multiples of the modulation frequency. Each cosine wave mixes with the corresponding component of the photocurrent to produce a signal after the low pass filter. The band-pass filter on the photodiode effectively eliminates all these higher frequency components in the photocurrent, so that only the fundamental cosine wave demodulation term is important.

Multiplying the component of the photocurrent at $\omega$ by $\cos \omega t$, we find

$$i_d = 4kE_2^2 E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin (\Omega t + \psi)}{\sqrt{1 + (\Omega / \omega_c)^2}} \cos^2 \omega t$$

The low-pass filter has a corner frequency that is much less than the modulation frequency. Thus, the component of $\cos^2 \omega t$ near DC will pass through, while the component at $2\omega$ will not, so that

$$i_o = 2kE_2^2 E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin (\Omega t + \psi)}{\sqrt{1 + (\Omega / \omega_c)^2}}$$

We define $H(f)$ as the transfer function from $x_0$ to $i_o$.

$$|H(f)| \equiv \left| \frac{i_o(f)}{x_0(f)} \right|$$

$$= 2k|E_2| E_+ \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{1}{\sqrt{1 + (\Omega / \omega_c)^2}}$$

where $\tilde{i_o}$ and $\tilde{x_0}$ denote the Fourier transforms of $i_o$ and $x_0$. 

III Noise

To quantify the noise performance of the interferometer, we must characterize the random process \( x(t) \) corresponding to the output in the absence of any signal. We shall use boldfaced symbols in our notation here to mean random processes and \( E\{\} \) to mean the expectation value or ensemble average. Early treatments of the shot noise assumed it was stationary and ignored the effect of the modulation of the photocurrent. Stationary noise is most conveniently represented using the one-sided power spectrum \( S_{xx}(f) \) of \( x(t) \). \( S_{xx}(f) \) is defined as the Fourier transform of the autocorrelation function \( R_{xx}(\tau) \) of \( x(t) \):

\[
R_{xx}(\tau) = E\{x(t+\tau)x(t)\}
\]

\[
S_{xx}(f) = 2 \int_{-\infty}^{\infty} R_{xx}(\tau) e^{2\pi if \tau} d\tau
\]

If \( x(t) \) is the input of a linear system, whose transfer function is \( H(f) \), and \( y(t) \) is the output, then

\[
S_{yy}(f) = |H(f)|^2 S_{xx}(f)
\]

The output \( i_0(t) \) of our model, in the absence of a signal, is not stationary since it fluctuates at the modulation frequency. However, it is cyclostationary, which is to say that for any \( t \), the statistics of \( i_0(t) \) are the same as those of \( i_0(t+T) \), where \( T \) is the period of modulation. In this situation if we define the average autocorrelation and power spectrum:

\[
\overline{R_{xx}}(\tau) = \frac{1}{T} \int_{t}^{t+T} R_{xx}(t' + \tau, t') dt'
\]

\[
\overline{S_{xx}}(f) = 2 \int_{-\infty}^{\infty} \overline{R_{xx}}(\tau) e^{2\pi if \tau} d\tau
\]

Then the relation

\[
\overline{S_{yy}}(f) = |H(f)|^2 \overline{S_{xx}}(f)
\]

holds true. Averaging in this way is equivalent to modeling the time reference or phase of the cyclostationary process as a random variable that is uniformly distributed over one cycle. In this case the phase-randomized process is stationary.\textsuperscript{21-22}

Our goal then is to calculate \( \overline{S_{i_0i_0}}(f) \), the average power spectrum of the interferometer output. We begin by finding \( \overline{S_{i_0i_0}}(f) \). The details of the derivation are in Appendix B; the result is:
This power spectrum has two sharp components, one at the modulation frequency, and one at its third harmonic, as well as a broadband component. Only the broadband component interests us, since it falls into the gravitational wave frequency band. The low-pass filter in our model of the detection system will leave this part of the noise spectrum unaffected and will attenuate the very high frequency components. Therefore,

\[
\overline{S}_{i,j}(f) = 3E_+^2 + E_{DC}^2 \\
+ (9E_+^4 + 6E_{DC}^2E_+^2 + E_{DC}^2 + 4E_+^2E_{DC}^2) \delta(2\pi f - \omega) \\
+ (E_+^4 + 4E_+^2E_{DC}^2) \delta(2\pi f - 3\omega)
\]  \hspace{1cm} (13)

Finally, the displacement noise in one test mass equivalent to shot noise is found by substituting from Eq. (8) and Eq. (14):

\[
\overline{S}_{i,j}(f) = 3E_+^2 + E_{DC}^2
\]  \hspace{1cm} (14)

IV Experiment

We have derived the differential mode displacement equivalent to shot noise in a power recycled interferometer with Fabry-Perot arm cavities. The 40-m interferometer on the Caltech campus provides us with an opportunity to compare the theory to measurement. From April, 1995 to August, 1996 the 40-m interferometer was operated in a recombined configuration, which is identical to the planned initial LIGO and VIRGO configurations without the recycling mirror.23 A recombined interferometer can be treated as a power recycled interferometer with recycling mirror transmission equal to one.

To compare our theoretical expression for shot noise with laboratory measurements we must determine the reflectivities and transmissions of the arm cavity mirrors as well as the fields present in the interferometer. The transmissions and losses of the mirrors can be obtained from in-situ measurements using the “ringdown” technique.24 This technique consists of building up a resonant field inside the cavity and then shutting off the power incident on the cavity. Observing the time scale of the exponential decay of the light leaking out of the cavity allows a calculation of the mirror parameters.25 The measured parameters are shown in Table 1.

The fields in the interferometer, however, are not available for direct measurement. What we instead measure is the DC voltage obtained by passing the antisymmetric port photocurrent through a known resistor. We record the minimum voltage when the interferometer is in lock \((V_{\text{min}})\) and the maximum voltage observed when the arm cavities are out of lock and the beam...
splitter is allowed to swing freely \( V_{\text{max}} \). The modulation depth (\( \Gamma \)) is measured using an optical spectrum analyzer. The fields are then found from

\[
E_2 = \sqrt{\frac{V_{\text{max}}}{Re}} J_0(\Gamma)
\]

\[
E_+ = \sqrt{\frac{V_{\text{max}}}{Re}} J_1(\Gamma) \sin \alpha
\]

\[
E_{\text{DC}} = \sqrt{\frac{V_{\text{min}}}{Re}} - 2E_+^2
\]

where \( R \) is the resistance in series with the photodiode and \( e \) is the charge of the electron in Coulombs. (Note that for comparison with experiment, we continue to write the fields in units of \( \sqrt{\text{photoelectrons/second}} \) as we have done for the theoretical expressions.)

To include the effect of light that is not mode matched properly into the arm cavities, we also measure the mode-matching fraction, \( M \).

\[
M = \frac{1 - R_{\text{arm}}}{1 - R_{\text{theory}}}
\]

where \( R_{\text{arm}} \) is the reflectivity of the arm cavities on resonance and \( R_{\text{theory}} \) is the theoretical reflectivity for a perfectly aligned cavity with the same mirror transmissions and losses. The mode-matching fraction affects the shot noise limit because only the light that could mode match into the cavities produces the signal. Mode matching does not affect the noise except as already accounted for in \( E_{\text{DC}} \). Thus the effective magnitude of \( E_2 \) and \( E_+ \) in the denominator of the shot noise expression (Eq. (15)) are reduced by \( \sqrt{M} \). So,

\[
S_{\Delta}(f)^{1/2} = \frac{\sqrt[3]{3E_+^2 + E_{\text{DC}}^2}}{2kME_+} \frac{(1 - r_3 r_4)^2}{T_3 r_4} \frac{\sqrt{1 + \left( \frac{2\pi f}{\omega_c} \right)^2}}{\sqrt{1 + \left( \frac{2\pi f}{\omega_c} \right)^2}}
\]

The parameters used in the shot noise calculation are collected in Table 2. The resulting curve is shown in Figure 4.

We would like to compare this calculated curve to an empirical measurement of the shot noise contribution to the gravitational wave signal (discussed below) and to the interferometer displacement spectrum taken at the time these measurements were done. The interferometer displacement spectrum can be obtained by monitoring a test point in the servo system electronics used to control differences in the lengths of the Fabry-Perot cavities when the interferometer is held on resonance with a dark fringe at the antisymmetric port. This signal can then be calibrated by actuating a mirror (mirror 4 of Figure 2) to produce known sinusoidal displacements at a frequency which is swept through the frequency range of interest. An empirical measurement of the shot noise contribution to the gravitational wave signal can be made by blocking the laser light and shining incandescent light on the antisymmetric photodiode such that the photocurrent is the
same as in normal operation. The gravitational wave readout equivalent to this shot noise can then be calibrated, provided the effect of the loop gain of servo systems controlling the interferometer is properly taken into account. With the interferometer in lock, the shot noise signal is suppressed by the differential mode loop gain. When the laser light is blocked, the differential mode loop is open and this suppression factor is no longer present.

The action of changing loop gain on various noise sources can be illustrated by a simple loop analysis. The differential mode servo loop, with the places where shot noise, dark noise of the photodiode and readout noise would sum in, is shown in Figure 5. The transfer functions from the noise inputs to the gravitational wave readout in the open loop case (when the laser light is blocked) are:

\[
\begin{align*}
\frac{x}{s}_{\text{open loop}} &= ABC \quad \frac{x}{n}_{\text{open loop}} = C \\
\frac{x}{s}_{\text{closed loop}} &= \frac{ABC}{1-L} \quad \frac{x}{n}_{\text{closed loop}} = C
\end{align*}
\]

With the loop closed during normal interferometer operation,

where the open loop gain is \( L = ABP \). Thus with the loop closed, shot noise and the dark noise of the photodiode are suppressed by \( 1/(1-L) \) relative to the open loop measurement while the readout noise is unaffected.

The complete measurement procedure for the empirical measurement of the shot noise limit shown in Figure 4 follows. The laser light is blocked after taking an interferometer displacement spectrum and the transfer function necessary for calibration. As a check of the readout noise, the input to the readout electronics is terminated in 50 \( \Omega \) and the power spectrum of the gravitational wave readout is recorded. After reconnecting the readout electronics, the power spectrum of the gravitational wave readout is recorded with no light on the antisymmetric photodiode. This is the dark noise spectrum and should be well above the level due to noise in the readout electronics, as it was in every case. Finally, the photodiode was illuminated with incandescent light to achieve the same photocurrent as is present during normal interferometer operation. The resulting power spectrum is the shot plus dark noise spectrum. The power spectrum of shot noise alone is recovered by quadrature subtraction of the dark noise. The shot noise power spectrum is then increased by 1 dB to reflect the fact that the measured fluctuations in the photocurrent from the photodiode are observed to be 1 dB greater for green laser light than for incandescent light producing the same DC photocurrent. (The origin of this effect is not understood\(^{26}\).) This spectrum is then divided by \( 1/(1-L) \), to account for the differential mode loop gain, and calibrated as usual to convert it into an equivalent amount of displacement noise.

Ideally, the shot noise power spectrum should be larger than the dark noise spectrum by a reasonable margin. In fact, for the measurement shown in the figure, there was only approximately a 3 dB margin which is why the resulting estimate for the shot noise contribution alone appears noisy.
V Conclusion

We have given a derivation of the shot noise limited sensitivity of a power recycled interferometer with Fabry-Perot arm cavities. The result can be compared with data from the 40-m interferometer operated in a recombined configuration without a recycling mirror. In particular an empirical measurement of the contribution of shot noise to the interferometer can be made, even in the presence of other noise sources. This empirical measurement of the shot noise contribution agrees with the calculation to within the uncertainties of the parameters in the calculation and in the calibration. Note that the interferometer was not limited by shot noise at any frequency. This was confirmed by attenuating the light leaving the antisymmetric port by 37.5% and directing light from an incandescent bulb onto the photodiode to raise the incident power by a factor of 3.2. We would expect a 7 dB increase in the interferometer displacement spectrum if it were limited by shot noise, but the largest increase seen anywhere in this frequency band was 4 dB. An extensive series of tests ruled out many potential sources for this excess noise, such as insufficient suppression of laser intensity or frequency noise. There was significant scattering from optics situated inside the beam splitter’s vacuum chamber and we were not able to extensively test whether this caused the noise excess. However the presence of this noise did not degrade our ability to confirm the shot noise contribution to the observed displacement spectrum as shown in Figure 4.

The methods used here for calculation of the shot noise contribution and for the empirical measurement of this contribution are quite general. They are directly applicable to the large-scale gravitational-wave detectors currently under construction for LIGO and VIRGO, and they can be readily adapted for other interferometer configurations.

Appendix A: Effect of Shaking an End Mirror

Here we derive the effect on $E_A$ of shaking mirror 4 at frequency $\Omega$ as mentioned in Section 2.4. To do this we will need to calculate the field reflected from the arm cavity, $E_5$, in terms of the incident field, $E_4$. This is done in two steps. First, we solve for $E_5$ given a small DC displacement of mirror 4. Then we generalize this result to frequencies in the gravitational wave band.

Consider a small displacement of mirror 4 away from the carrier resonance. Let $x_4 = 0$ on resonance so that $x_4 = x_0$ after the displacement. We define the arm cavity reflectivity away from resonance to be $r_{arm}(\phi)$ such that $E_5 = r_{arm}(\phi) E_4$, where

$$r_{arm}(\phi) = \frac{r_3 - (1 - L_3) r_4 e^{i\phi}}{1 - r_3 r_4 e^{i\phi}} \quad A-1$$

$$\phi = 2k x_4$$

Here $L_3$ is the loss associated with mirror 3 (assumed equal to the loss from mirror 4). Taylor expand $r_{arm}(\phi)$:
\[ E_5 = E_4 \left( r_{arm} \big|_{x_4 = 0} + \frac{dr_{arm}}{dx_4} \big|_{x_4 = 0} x_0 + \ldots \right) \tag{A-2} \]

Taking the derivative and noting \( d\phi/dx_4 = 2k \), for sufficiently small \( x_0 \) we can write

\[ E_5 = E_4 \left[ \frac{r_3 - (1 - L_3) r_4}{1 - r_3 r_4} - 2 i k \frac{T_3 r_4}{(1 - r_3 r_4)^2} x_0 \right] \tag{A-3} \]

Now let \( x_4 = x_0 \sin \Omega t \). Note that in the small amplitude limit we are considering, shaking the rear mirror at frequency \( \Omega \) phase modulates the light reflected from the mirror. This impresses sidebands on the reflected light at frequencies \( \Omega \) above and below the carrier frequency. The transmission of these sidebands from the rear mirror through the cavity is,

\[ t_{arm}(\phi) = \frac{t_3 e^{i\phi/2}}{1 - r_3 r_4 e^{i\phi}} \tag{A-4} \]

Now, \( \phi = 2 \omega l/c \) where \( \omega \) is the angular frequency of the light and \( l \) is the length of the cavity. The frequency of the light with the impressed sidebands from the mirror motion is \( \omega = \omega_0 \pm \Omega \) where \( \omega_0 \) is the carrier resonance frequency. If we assume \( \Omega \) is small compared to the cavity free spectral range, * we can approximate:

\[ \exp[i\phi] = \exp\left[2i(\omega_0 \pm \Omega)\frac{l}{c}\right] = \exp\left[i\Omega\frac{2l}{c}\right] \approx 1 \pm i\Omega\frac{2l}{c} \tag{A-5} \]

\[ t_{arm}(\phi) = \frac{t_3 \left(1 \pm i\Omega\frac{l}{c}\right)}{1 - r_3 r_4 \left(1 \pm i\Omega\frac{2l}{c}\right)} \tag{A-6} \]

This has a zero at angular frequency \( c/l \), which is twice the cavity free spectral range. This zero is well above the gravitational wave band and therefore not of interest. There is also a pole at angular frequency

\[ \omega_c = \frac{c}{2l} \frac{1 - r_3 r_4}{r_3 r_4} \tag{A-7} \]

This is the so-called cavity pole. It will typically be important and lie in the gravitational wave band.

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* The arm cavities for LIGO will have a free spectral range of 37.5 kHz, and for the 40-m interferometer it was 3.75 MHz.
We can now generalize from the DC case by noting that all the frequency dependence in the transfer function from $x_4$ to $E_5$ is contained in a single pole.

$$E_5 = E_4 \left[ \frac{r_3 - (1 - L_3) r_4}{1 - r_3 r_4} - 2 i k T_3 r_4 \frac{x_0 \sin(\Omega t + \psi)}{(1 - r_3 r_4)^2 \sqrt{1 + (\Omega/\omega_c)^2}} \right]$$  \hspace{1cm} \text{A-8}

where $\psi$ is a phase factor which is irrelevant for this analysis.

If we assume negligible losses and equal power transmission and reflection in the beam splitter,

$$E_4 = \frac{E_2}{\sqrt{2}} \quad E_6 = -\frac{E_2}{\sqrt{2}}$$  \hspace{1cm} \text{A-9}

The field at the antisymmetric port is

$$E_A = \frac{1}{\sqrt{2}} E_5 + \frac{1}{\sqrt{2}} E_7$$  \hspace{1cm} \text{A-10}

Now,

$$E_7 = E_6 \frac{r_5 - (1 - L_5) r_6}{1 - r_5 r_6}$$  \hspace{1cm} \text{A-11}

and

$$E_A = E_{DC} - i k E_2 \frac{T_3 r_4}{(1 - r_3 r_4)^2} \frac{x_0 \sin(\Omega t + \psi)}{\sqrt{1 + (\Omega/\omega_c)^2}}$$  \hspace{1cm} \text{A-12}

where $E_{DC}$ is the excess light at the antisymmetric port due to the imperfect matching of mirror parameters between the two arms or, more generally, any non-interfering sources of light on the photodetector.

**Appendix B: Average Power Spectrum of the Demodulator Output**

In this appendix, we derive the average power spectrum of the demodulator output, $\overline{S_{i_d i_d}(f)}$, from the time-averaged autocorrelation function, $\overline{R_{i_d i_d}(\tau)}$, using the methods discussed in reference 21. To calculate $\overline{R_{i_d i_d}(\tau)}$, we first find the expectation value of the photocurrent, $\overline{E\{i_p(t)\}}$, in the absence of any signal. In this case $E_A = E_{DC}$, thus from Eq. (4):
The total number of electrons having left the photodetector since some initial time \( t = 0 \) is modeled as a non-uniform Poisson process. A Poisson process \( q(t) \) is a random process which is constant except for unit increments at random points in time, \( t_i \). We label \( \lambda(t) \) the density of the points of \( t_i \). The term non-uniform applies if the density of points is a function of time. We identify \( \lambda(t) = E\{i_p(t)\} \). We write this as

\[
\lambda(t) = a + b \cos 2\omega t + c \sin 2\omega t
\]

The photodetector output current is then a random process that is the derivative of a Poisson process. This is called a process of Poisson impulses.

\[
i_p(t) = \frac{dq(t)}{dt} = \sum_i \delta(t - t_i)
\]

The autocorrelation of a non-uniform Poisson process is:

\[
R_{qq}(t_1, t_2) = \begin{cases} 
\int_0^{t_2} \lambda(t) \, dt \left[ 1 + \int_0^{t_1} \lambda(t) \, dt \right] & t_1 > t_2 \\
\int_0^{t_1} \lambda(t) \, dt \left[ 1 + \int_0^{t_2} \lambda(t) \, dt \right] & t_2 > t_1
\end{cases}
\]

The autocorrelation of the derivative of a random process is given by

\[
R_{q'q'}(t_1, t_2) = \frac{\partial^2 R_{xx}(t_1, t_2)}{\partial t_1 \partial t_2}
\]

Since \( i_p(t) = x'(t) \) we have only to substitute into the equation above to find the autocorrelation for the photocurrent. So,
However, there is a discontinuity in the derivative at \( t_1 = t_2 \). Thus,
\[
R_{i_p,i_p}(t_1, t_2) = \frac{\partial^2 R_{qq}(t_1, t_2)}{\partial t_1 \partial t_2}
\]
\[
= \begin{cases} 
\lambda(t_1) \lambda(t_2) & t_1 > t_2 \\
\lambda(t_2) \lambda(t_1) & t_2 > t_1
\end{cases}
\]

We can use this result to find the time-averaged autocorrelation of the demodulator output \( i_d(t) = i_p(t) \cos \omega t \):
\[
R_{i_d,i_d}(t + \tau, t) = E\{i_p(t + \tau) \cos \omega(t + \tau) \ i_p(t) \cos \omega t \}
\]
\[
= E\{i_p(t + \tau) \ i_p(t)\} \cos \omega(t + \tau) \cos \omega t
\]
\[
= [\lambda(t + \tau) \lambda(t) + \lambda(t + \tau) \delta(t) \cos \omega(t + \tau) \cos \omega t
\]

\[
\overline{R_{i_d,i_d}}(\tau) = \frac{1}{T} \int_{0}^{T} R_{i_d,i_d}(t + \tau, t) \ dt
\]
\[
= \frac{1}{T} \int_{0}^{T} [\lambda(t + \tau) \lambda(t) + \lambda(t + \tau) \delta(t) \cos \omega(t + \tau) \cos \omega t \ dt
\]

where \( T \) is the modulation period.

To find the average power spectrum, we take the Fourier transform of the average autocorrelation:
\[
\overline{S_{i_d,i_d}}(f) = 2 \int_{-\infty}^{\infty} \overline{R_{i_d,i_d}}(\tau) e^{2\pi i f \tau} d\tau
\]

We will evaluate the two terms in Eq. B-9 one at a time. The first term gives
Substituting this into Eq. B-10 yields

\[
\frac{1}{T} \int_0^T \lambda(t + \tau) \lambda(t) \cos \omega(t + \tau) \cos \omega t \, dt
\]

\[
= \frac{1}{T} \int_0^T [(a + b \cos 2 \omega t + c \sin 2 \omega t) \cos \omega(t + \tau) \cos \omega t \, dt
\]

\[
= \frac{1}{2} \left[ \left( a^2 + ab + \frac{1}{4}(b^2 + c^2) \right) \cos \omega \tau + \frac{1}{4}(b^2 + c^2) \cos 3 \omega \tau \right]
\]

Substituting this into Eq. B-10 yields

\[
2 \int_{-\infty}^{\infty} \frac{1}{2} \left[ \left( a^2 - 2ab + \frac{1}{4}(b^2 + c^2) \right) \cos \omega \tau + \frac{1}{4}(b^2 + c^2) \cos 3 \omega \tau \right] e^{2\pi i f \tau} \, d\tau
\]

\[
= \left( a^2 + ab + \frac{1}{4}(b^2 + c^2) \right) \delta(2\pi f - \omega) + \frac{1}{4}(b^2 + c^2) \delta(2\pi f - 3\omega)
\]

To evaluate the second term we will reverse the order of integration:

\[
\frac{2}{T} \int_{-\infty}^{\infty} \int_0^T \lambda(t + \tau) \delta(\tau) \cos \omega(t + \tau) \cos \omega t \, dt \, e^{2\pi i f \tau} \, d\tau
\]

\[
= \frac{2}{T} \int_0^T \lambda(t + \tau) \delta(\tau) \cos \omega(t + \tau) \cos \omega t \, e^{2\pi i f \tau} \, d\tau \, dt
\]

\[
= \frac{2}{T} \int_0^T \lambda(t) \cos^2 \omega t \, dt
\]

\[
= \frac{2}{T} \int_0^T (a + b \cos 2 \omega t + c \sin 2 \omega t) \cos^2 \omega t \, dt
\]

\[
= a + \frac{b}{2}
\]

Therefore, the average power spectrum of the demodulator output is
\[
\overline{S_{id}}(f) = 3E_+^2 + E_{DC}^2 \\
+ (9E_+^4 + 6E_{DC}^2E_+^2 + E_{DC}^2 + 4E_{2\times}^2E_{DC}^2) \delta(2\pi f - \omega) \\
+ (E_+^4 + 4E_{2\times}^2E_{DC}^2) \delta(2\pi f - 3\omega)
\]

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10. K.S. Thorne, in *300 Years of Gravitation*, ed. S.W. Hawking and W. Israel, equation 115 (Cambridge University


11. J.Y. Vinet, B. Meers, C.N. Man and A. Brillet, “Optimization of long-baseline optical interferometers for gravita-

12. D. Shoemaker, P. Fritschel, J. Giaime, N. Christensen, and R. Weiss, “Prototype Michelson interferometer with


13. S. Whitcomb, “Shot Noise in the Caltech 40 m Interferometer”, LIGO internal document, LIGO-T850002-00-D

(1985).


26. Others have observed that illuminating the entire surface of a photodiode can cause such an effect, which can be eliminated if only the active region is illuminated. (David H. Shoemaker, private communication, October, 1999.) In our case the laser beam illumination was well within the active region whereas the incandescent light illuminated the entire photodiode. Unfortunately we did not try changing the collimation of the incandescent light.

27. Papoulis, p. 291, equation (10-17).

### Table 1: Parameters for the 40-m Interferometer

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Value</th>
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<tr>
<td>Mirror (power) transmissions</td>
<td>$T_2$</td>
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<td></td>
<td>$T_3$</td>
<td>280 ppm</td>
</tr>
<tr>
<td></td>
<td>$T_5$</td>
<td>300 ppm</td>
</tr>
<tr>
<td></td>
<td>$T_4, T_6$</td>
<td>12 ppm</td>
</tr>
<tr>
<td>Loss in each mirror</td>
<td>$L_3, L_4$</td>
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<td></td>
<td>$L_5, L_6$</td>
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</tr>
<tr>
<td>Asymmetry</td>
<td>$\delta$</td>
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</tr>
<tr>
<td>Modulation frequency</td>
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</tr>
<tr>
<td>Modulation index</td>
<td>$\Gamma$</td>
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</tr>
<tr>
<td>Contrast defect</td>
<td>$1 - C$</td>
<td>0.03</td>
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</table>
Table 2: Parameters Used in Shot Noise Calculation

<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
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<tbody>
<tr>
<td>$V_{\text{max}}$</td>
<td>1.1 V</td>
</tr>
<tr>
<td>$V_{\text{min}}$</td>
<td>20 mV</td>
</tr>
<tr>
<td>$R$</td>
<td>50 Ω</td>
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<tr>
<td>$\Gamma$</td>
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<tr>
<td>$M$</td>
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<tr>
<td>$\alpha$</td>
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<td>$T_3$</td>
<td>280 ppm</td>
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<tr>
<td>$r_3$</td>
<td>0.999805</td>
</tr>
<tr>
<td>$r_4$</td>
<td>0.999938</td>
</tr>
</tbody>
</table>

Figure 1: Power recycled interferometer with Fabry-Perot arm cavities.
Figure 2: Recycled interferometer with mirrors and optical fields labelled.

Figure 3: Detection system for the antisymmetric port light.
Figure 4: Calculated shot noise contribution to interferometer displacement spectrum (dashed), with empirical measurement of shot noise contribution (dotted) and interferometer displacement spectrum taken shortly before on January 10, 1996 (solid).
Figure 5: Differential mode servo loop with shot noise, dark noise and readout noise inputs.